

# NONCOMMUTATIVE VITALI-HAHN-SAKS THEOREM HOLDS PRECISELY FOR FINITE $W^*$ -ALGEBRAS.

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ABSTRACT. It is shown that the bona fide generalization of the Vitali-Hahn-Saks Theorem to von Neumann algebras is possible if, and only if, the algebra is finite. This settles the problem on the noncommutative Vitali-Hahn-Saks Theorem completely and provides new means of characterizing finite von Neumann algebras.

## 1. INTRODUCTION AND PRELIMINARIES

The Vitali-Hahn-Saks Theorem is one of the fundamental results of measure theory. Let  $K$  be a set of (scalar-valued) measures on a  $\sigma$ -field  $\mathcal{A}$  of measurable sets. Suppose that  $K$  is relatively compact in the topology of pointwise convergence on the elements of  $\mathcal{A}$  and that every  $\varphi \in K$  is absolutely continuous with respect to some fixed positive measure  $\psi$  on  $\mathcal{A}$ . In its classical form, the Vitali-Hahn-Saks Theorem asserts that  $K$  is uniformly absolutely continuous with respect to  $\psi$ .

The generalization of the Vitali-Hahn-Saks Theorem to von Neumann algebras has recently received a great deal of attention. Deep classical results of Aarnes and Akemann [1, 2] have been considerably extended to  $C^*$ -algebras and von Neumann algebras by Brooks, Saitô and Wright in a series of remarkable papers [3, 4, 5]. However, in order to obtain uniform absolute continuity, in these papers it is assumed that  $K$  is pointwise *strongly* absolutely continuous with respect to  $\psi$ . Let us recall that a normal functional  $\varphi$  on a von Neumann algebra  $M$  is absolutely continuous with respect to a normal positive functional  $\psi$  on  $M$  if  $\varphi(p) = 0$  whenever  $p$  is a projection in  $M$  with  $\psi(p) = 0$ . On the other hand,  $\varphi$  is said to be strongly absolutely continuous with respect to  $\psi$  if both the absolute value  $|\varphi_h|$  of the hermitian part of  $\varphi$  and the absolute value  $|\varphi_{ah}|$  of the antihermitian part of  $\varphi$  are absolutely continuous with respect to  $\psi$ . Strong absolute continuity is much more stringent than absolute continuity; particularly if we consider vector-valued measures. The two notions coincide if the algebra is abelian or when all measures concerned are positive. Therefore a natural question has erupted on whether the Vitali-Hahn-Saks Theorem holds without assuming strong absolute continuity in the hypothesis. In [7] we showed the 'genuine' form of the Vitali-Hahn-Saks Theorem can be obtained provided the

control measure is faithful. In this paper we settle completely the status of the Vitali-Hahn-Saks Theorem for von Neumann algebras. We show that the direct generalization is possible for vector-valued measures on finite von Neumann algebras. On the other hand, we prove that if the algebra is infinite, the Vitali-Hahn-Saks Theorem fails, even if we restrict to scalar-valued measures. This new measure-theoretic characterization of finite von Neumann algebras complements the wide range of hitherto known topological, functional-analytic, and lattice-theoretic characterizations of finiteness in the Murray-von Neumann comparison theory (see e.g. [9, 10, 11]). Since for positive measures the Vitali-Hahn-Saks Theorem holds irrespectively of the von Neumann algebra (see e.g. [7, Theorem 4.6]), the results obtained in this paper exhibit the delicate interplay that exists between the measure and its absolute value in the noncommutative situation. Pursuing this matter further, we show that for each infinite von Neumann algebra there is a weakly relatively compact subset  $K$  in the predual such that the Vitali-Hahn-Saks Theorem holds for  $K$ , but not for the set  $|K|$  (where  $|K|$  denotes the set of absolute values of the functionals of  $K$ ). This extends a classical result of Saitô [10], who proved that a von Neumann algebra  $M$  is finite if, and only if, the following condition holds: A subset  $K$  of the predual of  $M$  is weakly relatively compact exactly when  $|K|$  is weakly relatively compact. In this connection we have obtained further descriptions of finite von Neumann algebras.

Let us recall basic facts and fix the notation. Our standard reference for operator algebras is [11]. For a normed space  $F$  we shall use the symbol  $F_1$  for its closed unit ball. The symbol  $B(\mathcal{H})$  will be reserved for the algebra of all bounded operators acting on a Hilbert space  $\mathcal{H}$ . Throughout the paper,  $M$  will stand for a von Neumann algebra. The symbols  $M_*$  and  $M_*^+$  shall denote the predual and the positive part of the predual, respectively. If  $M$  acts on a Hilbert space  $\mathcal{H}$  and  $\eta, \xi \in \mathcal{H}$ , we shall denote by  $\omega_{\eta, \xi}$  the linear functional  $\omega_{\eta, \xi}(x) = (x\eta, \xi)$  ( $x \in M$ ). We also set  $\omega_\eta = \omega_{\eta, \eta}$ . If  $\varphi \in M_*$  we shall denote by  $|\varphi|$  the absolute value of the functional  $\varphi$  (see e.g. [11] for more details). Let  $P(M)$  denote the projection lattice of  $M$ . Two projections  $p, q \in P(M)$  are called orthogonal if  $pq = 0$ .

We denote by  $\sigma(M, M_*)$  the weak\*-topology on  $M$ , i.e. the weakest topology compatible with the duality  $\langle M_*, M \rangle$ . The strongest topology on  $M$  compatible with this duality (the Mackey topology) is denoted by  $\tau(M, M_*)$ . We recall that the  $\tau(M, M_*)$  topology on  $M$  coincides with the topology of uniform convergence on weakly relatively compact subsets of  $M_*$ . Lying between these topologies we have the  $\sigma$ -strong topology  $s(M, M_*)$  determined by the family of seminorms  $\{\rho_\omega \mid \omega \in M_*^+\}$  where  $\rho_\omega(x) = \omega(x^*x)^{1/2}$ ; and the  $\sigma$ -strong\* topology  $s^*(M, M_*)$  determined by the family of the seminorms  $\{\rho_\omega, \rho_\omega^* \mid \omega \in M_*^+\}$ , where

$\rho_\omega^*(x) = \omega(xx^*)^{1/2}$ . Let us recall that on bounded parts of  $M$  the  $\sigma$ -strong\* topology coincides with the Mackey topology.

Let  $X$  be a locally convex vector space. A *completely additive* measure  $\mu : P(M) \rightarrow X$  is a map satisfying  $\mu\left(\sum_{p \in \Gamma} p\right) = \sum_{p \in \Gamma} \mu(p)$ , whenever  $\Gamma$  is a set of pairwise orthogonal projections. By the symbol  $B_{ca}(M, X)$  we shall denote the set of all bounded linear maps of  $M$  into  $X$  which restrict to completely additive measures on  $P(M)$ . (If  $X = \mathbb{C}$ , then  $B_{ca}(M, X)$  reduces to the predual of  $M$ .) By the vector Gleason theorem [6] there is a one-to-one correspondence between bounded completely additive measures on  $P(M)$  and operators in  $B_{ca}(M, X)$  (see [7, 8] for more details).

A subset  $K \subset B_{ca}(M, X)$  is said to be *pointwise absolutely continuous* with respect to  $\psi \in M_*^+$  (in symbols  $K \ll_p \psi$ ) if for every  $T \in K$  and neighbourhood  $U$  of  $0 \in X$ , there is a  $\delta > 0$  such that  $Tp \in U$  whenever  $p \in P(M)$  and  $\psi(p) < \delta$ . For every  $T \in B_{ca}(M, X)$  we write  $T \ll \psi$  if  $\{T\} \ll_p \psi$ . A subset  $K \subset B_{ca}(M, X)$  is said to be *uniformly absolutely continuous* with respect to  $\psi \in M_*^+$  (in symbols  $K \ll_u \psi$ ) if for every neighbourhood  $U$  of  $0 \in X$ , there is a  $\delta > 0$  such that  $Tp \in U$  for every  $T \in K$  whenever  $p \in P(M)$  and  $\psi(p) < \delta$ . (For more details on the interrelations between various concepts of absolute continuity see [7].)

Given a subset  $K \subset M_*$  we define  $K_p = \{\psi \in M_*^+ \mid K \ll_p \psi\}$  and  $K_u = \{\psi \in M_*^+ \mid K \ll_u \psi\}$ . Of course,  $K_u \subset K_p$ . A subset  $K \subset M_*$  is said to have the *Vitali-Hahn-Saks property* (VHS-property in short) if  $K_p \neq \emptyset$  and  $K_p = K_u$ . A deep result of Akemann states that  $K_u$  is nonempty if  $K \subset M_*$  is weakly relatively compact [2]. If  $K \subset M_*$  is bounded and enjoys the Vitali-Hahn-Saks property, then it has a control measure and therefore  $K$  is weakly relatively compact. However, it will follow from our discussion that in the predual of any infinite algebra there are weakly relatively compact subsets that do not have the Vitali-Hahn-Saks property.

## 2. RESULTS

Before giving the proof of the Vitali-Hahn-Saks Theorem for finite algebras we recall that the following three conditions are equivalent: (i)  $M$  is finite; (ii) The  $*$ -operation is  $\sigma$ -strongly continuous; (iii) On bounded parts of  $M$  the  $(\sigma)$ -strong topology agrees with the Mackey topology [11, p. 333, Exercise 5].

**2.1. Theorem.** *Let  $M$  be a finite von Neumann algebra and  $K \subset B_{ca}(M, X)$  a relatively compact set in the topology of pointwise convergence on elements of  $M$ . Let  $\psi \in M_*^+$ . If  $K$  is pointwise absolutely continuous with respect to  $\psi$ , then  $K$  is uniformly absolutely continuous with respect to  $\psi$ .*

*Proof.* First we shall prove this theorem for the scalar case, i.e. when  $X = \mathbb{C}$ . In this case  $K$  is a weakly relatively compact subset of  $M_*$ . Let  $(e_k)$  be a sequence of projections such that  $\psi(e_k) \rightarrow 0$ . Denote by  $s(\psi)$  the support projection of  $\psi$  and let  $N = s(\psi)Ms(\psi)$ . Since  $\psi$  is faithful on  $N$ ,  $s(\psi)e_k s(\psi) \rightarrow 0$  in the  $s(N, N_*)$  topology (and therefore in the  $\sigma(N, N_*)$  topology) [11, p. 148, Prop. 5.3]. On  $N$  the  $\sigma(N, N_*)$  topology coincides with the relativized  $\sigma(M, M_*)$  topology. Thus it follows that  $e_k s(\psi) \rightarrow 0$  in the  $s(M_*, M)$  topology. Since  $M$  is finite  $s(\psi)e_k = (e_k s(\psi))^* \rightarrow 0$  in the  $s(M_*, M)$  topology. This implies that

$$s(\psi)e_k s(\psi), (1 - s(\psi))e_k s(\psi), s(\psi)e_k(1 - s(\psi)) \rightarrow 0$$

in the  $s(M, M_*)$  topology - and therefore in the  $\tau(M, M_*)$  topology. Moreover, since every  $\varphi \in K$  is absolutely continuous with respect to  $\psi$  we have  $\varphi((1 - s(\psi))e_k(1 - s(\psi))) = 0$  for every  $k$  [7, Proposition 2.2]. Combining we get that as  $k \rightarrow \infty$

$$\begin{aligned} \sup_{\varphi \in K} |\varphi(e_k)| &\leq \sup_{\varphi \in K} |\varphi(s(\psi)e_k s(\psi))| + \sup_{\varphi \in K} |\varphi((1 - s(\psi))e_k s(\psi))| \\ &\quad + \sup_{\varphi \in K} |\varphi(s(\psi)e_k(1 - s(\psi)))| \rightarrow 0. \end{aligned}$$

Let us return to the general vector case. Let  $U$  be a convex, closed, circled neighbourhood of zero in  $X$  with the polar  $U^0 = \{f \in X^* \mid |f(x)| \leq 1\}$ . Then the set

$$K_U = \{f \circ T \mid f \in U^0, T \in K\}$$

is a weakly relatively compact subset of  $M_*$  by [7, Theorem 3.3]. By the previous part of the proof  $K_U \ll_u \psi$ . In other words, there is a  $\delta > 0$  such that  $|f(T(p))| \leq 1$  for all  $f \in U^0$  and  $T \in K$ , whenever  $p \in P(M)$  and  $\psi(p) < \delta$ . Therefore, if  $\psi(p) < \delta$ , then  $T(p) \in U$  for all  $T \in K$  by the Bipolar Theorem.  $\square$

**2.2. Proposition.** *Let  $M$  be an infinite von Neumann algebra. Then there is a normal state  $\psi$  on  $M$  and a sequence  $(\varphi_k)$  of normal functionals on  $M$  such that each  $\varphi_k$  is absolutely continuous with respect to  $\psi$ ,  $(\varphi_k)$  converges to zero pointwise on  $M$ , but  $(\varphi_k)$  is not uniformly absolutely continuous with respect to  $\psi$ .*

*Proof.* Suppose that  $M$  acts on a Hilbert space  $H$ . Since  $M$  has nonzero properly infinite part, there is a sequence  $(e_n)$  of mutually orthogonal nonzero projections in  $M$  which are pairwise equivalent. Therefore, there is a sequence  $(u_k)$  of partial isometries in  $M$  such that  $u_k^* u_k = e_1$ ,  $u_k u_k^* = e_k$  for each  $k \geq 2$ . Let us choose a unit vector  $\xi_1 \in e_1(H)$  and put  $\xi_k = u_k \xi_1$ ,  $k \geq 2$ . The sequence  $(\xi_k)$  is orthonormal. Let us further define

$$\psi = \sum_{n=2}^{\infty} \frac{1}{2^n} \omega_{\xi_n} \quad \text{and} \quad \varphi_k = \omega_{\xi_k, \xi_1} \quad (k \geq 2).$$

It is clear that  $\varphi_k \rightarrow 0$  weakly. Let  $p \in P(M)$  with  $\psi(p) = 0$ . Then, for each  $k \geq 2$ ,  $p\xi_k = 0$ , and so  $\varphi_k(p) = (p\xi_k, \xi_1) = 0$ . Whence, the sequence  $(\varphi_k)_{k \geq 2}$  is pointwise absolutely continuous with respect to  $\psi$ . For each  $k \geq 2$ , define  $h_k = \frac{1}{2}(e_1 + e_k + u_k + u_k^*)$ . Observe that  $h_k$  is a projection lying underneath  $e_1 + e_k$ . We can compute

$$\begin{aligned} \psi(h_k) &= \sum_{n=2}^{\infty} \frac{1}{2^n} \omega_{\xi_n}(h_k) = \sum_{n=2}^{\infty} \frac{1}{2^n} (h_k \xi_n, \xi_n) = \\ &= \frac{1}{2^k} (h_k \xi_k, \xi_k). \end{aligned}$$

Thus  $\psi(h_k) \leq \frac{1}{2^k} \rightarrow 0$  as  $k \rightarrow \infty$ . On the other hand  $\varphi_k(h_k) = (h_k \xi_k, \xi_1) = \frac{1}{2}$ , whenever  $k \geq 2$ . Whence, for each fixed  $n$

$$\sup_k |\varphi_k(h_n)| \geq \frac{1}{2},$$

and so the sequence  $(\varphi_k)$  is not uniformly absolutely continuous with respect to  $\psi$ .  $\square$

**2.3. Theorem.** *The following conditions are equivalent*

- (i)  *$M$  is finite*
- (ii) *Let  $K \subset M_*$  be a weakly relatively compact set and  $\psi \in M_*^+$ . If  $K \ll_p \psi$ , then  $K \ll_u \psi$*

Saitô showed that if  $M$  is a finite von Neumann algebra and  $K \subset M_*$  is weakly relatively compact, then  $|K|$  is also weakly relatively compact [10, Theorem 1]. Moreover, he proved that this property of the predual characterizes finite algebras completely. (Of course, if  $|K|$  is weakly relatively compact, then  $K$  is also weakly relatively compact irrespective of whether the algebra is finite or not.) Theorem 2.1 and [2, Theorem III.9] (see also [11, p.149, Theorem 5.4]) tells us that for finite algebras, weakly relatively compact sets in the predual coincide with the bounded sets enjoying the Vitali-Hahn-Saks property. Combining with the result of Saitô we conclude that for finite algebras:

$K$  has the VHS-property if, and only if,  $|K|$  has the VHS-property, for every bounded subset  $K$  of the predual. In Theorem 2.6 below we show that for  $\sigma$ -finite algebras both implications imply finiteness of the algebra. Let us first prove the following lemmas.

**2.4. Lemma.** *Let  $M$  be a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$  and let  $(\eta_i) \subset \mathcal{H}$  be a (norm) convergent sequence of vectors. Then, the set*

$$\{\omega_{\eta_i, \xi} \mid i \in \mathbb{N}, \xi \in \mathcal{H}_1\} \subset M_*$$

*is weakly relatively compact.*

*Proof.* Let  $\mathcal{H}^1$  and  $\mathcal{H}^2$  be two copies of  $\mathcal{H}$ . Equip  $\mathcal{H}_1^1$  with the norm topology and  $\mathcal{H}_1^2$  with the weak topology. Consider the Cartesian product  $\mathcal{H}_1^1 \times \mathcal{H}_1^2$ . One can easily check that the function  $\Gamma : (\eta, \xi) \in$

$\mathcal{H}_1^1 \times \mathcal{H}_1^2 \mapsto \omega_{\eta, \xi} \in M_*$  is continuous (where  $M_*$  is equipped with the weak topology). Therefore,  $\Gamma$  maps relatively compact subsets of  $\mathcal{H}_1^1 \times \mathcal{H}_1^2$  into weakly relatively compact subsets of  $M_*$ . But, observe that the set

$$\{(\eta_i, \xi) \mid i \in \mathbb{N}, \xi \in \mathcal{H}_1^2\} \subset \mathcal{H}_1^1 \times \mathcal{H}_1^2$$

is relatively compact.  $\square$

**2.5. Lemma.** *Let  $M$  be a von Neumann algebra and let  $A$  be a von Neumann subalgebra of  $M$  sharing the same unit  $\mathbf{1}$  of  $M$ . For any  $\varphi \in M_*$ , denote by  $\widehat{\varphi}$  the restriction of  $\varphi$  to  $A$ . If  $\|\varphi\| = \|\widehat{\varphi}\|$  for some  $\varphi \in M_*$ , then  $|\widehat{\varphi}| = |\varphi|$ .*

*Proof.* We observe that

$$\|\widehat{|\varphi|}\| = |\widehat{\varphi}|(\mathbf{1}) = |\varphi|(\mathbf{1}) = \|\varphi\| = \|\widehat{\varphi}\|.$$

Moreover,

$$|\varphi(x)|^2 \leq \|\varphi\| |\varphi|(xx^*) \quad \text{for all } x \in M,$$

which implies that  $|\widehat{\varphi}(x)|^2 \leq \|\widehat{\varphi}\| |\widehat{\varphi}|(xx^*)$  for all  $x \in A$ . Thus result follows from [11, p. 143, Proposition 4.6].  $\square$

**2.6. Theorem.** *Let  $M$  be a  $\sigma$ -finite von Neumann algebra. The following statements are equivalent:*

- (i)  $M$  is finite;
- (ii)  $|K|$  has the Vitali-Hahn-Saks property implies that  $K$  has the Vitali-Hahn-Saks property, for all bounded  $K \subset M_*$ .
- (iii)  $K$  has the Vitali-Hahn-Saks property implies that  $|K|$  has the Vitali-Hahn-Saks property, for all bounded  $K \subset M_*$ ;

*Proof.* We know that if  $M$  is finite, then weakly relative compact sets of the predual coincide with the bounded sets enjoying the Vitali-Hahn-Saks property. Therefore (i)  $\implies$  (ii) and (i)  $\implies$  (iii) by [10, Theorem 1].

Suppose now that  $M$  is not finite. Consider the sequence  $\varphi_k = \omega_{\xi_k, \xi_1}$ ,  $k \geq 2$ , constructed in the proof of Proposition 2.2. It was shown that the sequence  $K = (\varphi_k)_{k \geq 2}$  does not have the Vitali-Hahn-Saks property. On the other hand,  $|K| = \{\omega_{\xi_1}\}$  and therefore it has the Vitali-Hahn-Saks property. Whence, (ii)  $\implies$  (i).

It remains to show that (iii)  $\implies$  (i). For this, we shall suppose that  $M$  is not finite and construct a sequence  $K \subset M_*$  such that  $K$  has the Vitali-Hahn-Saks property while  $|K|$  has not the Vitali-Hahn-Saks property. Since  $M$  has a nonzero properly infinite direct summand, we can assume that  $M$  is properly infinite, and thereby isomorphic to the algebra  $B(\mathcal{H}) \otimes N$  where  $\mathcal{H}$  is a separable infinite-dimensional Hilbert space, and  $N$  is a  $\sigma$ -finite von Neumann algebra.  $N$  admits a faithful normal representation  $\{\pi, \mathcal{K}\}$  with a separating and cyclic (unit) vector

$\eta \in \mathcal{K}$ . Therefore, upon replacing  $M$  by  $(I \otimes \pi)M$ , we can assume that  $M$  acts on  $\mathcal{H} \otimes \mathcal{K}$ .

Let  $(\eta_i)$  be a total and convergent sequence of unit vectors in  $\mathcal{H}$ . For each  $i \in \mathbb{N}$ , let  $\bar{\eta}_i = \eta_i \otimes \eta$ . We show that the sequence  $(\bar{\eta}_i)$  is separating for  $M$ . For this, it is enough to show that  $(\bar{\eta}_i)$  is cyclic for  $M'$ . This follows by the commutation theorem for tensor products after observing that

$$[M'\{\bar{\eta}_i\}_i] = [\mathbb{C} \otimes N'\{\eta_i \otimes \eta\}_i] = \mathcal{H} \otimes \mathcal{K}.$$

Consequently, the set in the predual

$$K = \{\omega_{\bar{\eta}_i, \bar{\xi}} \mid i \in \mathbb{N}, \bar{\xi} \in (\mathcal{H} \otimes \mathcal{K})_1\}$$

is separating for  $M$ . In view of Lemma 2.4,  $K$  is also weakly relatively compact. Since  $M$  is  $\sigma$ -finite,  $M$  admits a faithful normal state. Thus  $K_p \neq \emptyset$ . Moreover, since  $K$  is separating for  $M$ , every  $\psi \in K_p$  is necessarily faithful. By [7, Theorem 4.1] it follows that  $K_p = K_u$ , i.e.  $K$  enjoys the Vitali-Hahn-Saks property.

We show that  $|K|$  does not have the VHS-property. Let  $(\xi_j)$  be an orthonormal basis of  $\mathcal{H}$  and let  $\bar{\xi}_j = \xi_j \otimes \eta$ . We show that the set

$$\{|\omega_{\bar{\eta}_i, \bar{\xi}_j}| \mid i, j \in \mathbb{N}\}$$

is not weakly relatively compact. If we identify  $B(\mathcal{H})$  with  $B(\mathcal{H}) \otimes \mathbf{1}_N$ , we can assume that  $B(\mathcal{H})$  is a von Neumann subalgebra of  $M$ , sharing the same unit of  $M$ . The restriction to  $B(\mathcal{H})$  of  $\omega_{\bar{\eta}_i, \bar{\xi}_j}$  is  $\omega_{\eta_i, \xi_j}$ . Since  $1 = \|\omega_{\eta_i, \xi_j}\| = \|\omega_{\bar{\eta}_i, \bar{\xi}_j}\|$ , by Lemma 2.5, it follows that the restriction to  $B(\mathcal{H})$  of  $|\omega_{\bar{\eta}_i, \bar{\xi}_j}|$  is  $|\omega_{\eta_i, \xi_j}|$ . But observe that  $|\omega_{\eta_i, \xi_j}| = \omega_{\xi_j}$ . However, the set  $\{\omega_{\xi_j} \mid j \in \mathbb{N}\}$  is not weakly relatively compact. This means that  $|K|$  cannot have the Vitali-Hahn-Saks property.  $\square$

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## REFERENCES

- [1] J. Aarnes, *The Vitali-Hahn-Saks Theorem for von Neumann algebras*, Math. Scand. **18** (1966), 87–92.
- [2] C. A. Akemann, *The dual space of an operator algebra*, Trans. Amer. Math. Soc. **126** (1967), 286–302.
- [3] J. K. Brooks and J. D. M. Wright, *Convergence in the dual of a  $\sigma$ -complete  $C^*$ -algebra*, J. Math. Appl. **294** (2004), 141–146.
- [4] J. K. Brooks, K. Saitô and J. D. M. Wright, *When absolute continuity on  $C^*$ -algebras is uniform*, Quart. J. Math. **55** (2004), 31–40.
- [5] J. K. Brooks, K. Saitô and J. D. M. Wright, *Operators on  $\sigma$ -complete  $C^*$ -algebras*, Quart. J. Math. **56** (2005), 301–310.

- [6] L. J. Bunce and J. D. M. Wright, *The Mackey-Gleason Problem*, Bull. Amer. Math. Soc. **26** No. 2 (1992), 288–293.
- [7] E.Chetcuti and J.Hamhalter, *Vitali-Hahn-Saks Theorem for vector measures on operator algebras*, Quart. J. Math. **57** (2006), 479-493.
- [8] J. Hamhalter, *Quantum Measure Theory*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.
- [9] J.Hamhalter, *Spectral Order of Operators and Range Projections*, J. Math. Appl., 2007, to appear.
- [10] K.Saitô: *On the preduals of  $W^*$ -algebras*, Tôhoku Math. J. (2) **19**, (1967), 324-331.
- [11] M. Takesaki, *Theory of Operator Algebras I*, Berlin, Heidelberg, New York, Springer, 1979.

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